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Asymptotic Behavior of nth Order Impulsive Differential Equations

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Abstract: This paper devotes to the asymptotic behavior of all solutions of nth order impulsive differential equations. Based on impulsive differential inequality, boundedness and zero tendency of every solution for nth order impulsive differential equations are obtained. In addition, we derive globally uniformly exponential stability of every solution under Lyapunov function and impulsive technique, and these results are extend to nth order differential equations with periodic coefficient and periodic impulse. Meanwhile, an example with simulations are provided to verify the conclusion.

Keywords: Asymptotic, nth Order, Impulsive Differential Equation

1. Introduction

Due to the abrupt changes at certain moments, impulsive effects are common phenomena in natural world. Such phenomena are described by impulsive differential equations which have been used efficiently in modelling many real world problems that arise in the fields such as medicine, electronics, and network [1–7]. The behavior of all solutions is an important issue of impulsive differential equations. Therefore, much effort has been made to investigate the oscillation criteria and the asymptotic behavior of all solutions of impulsive differential equations [8–16]. For instance, in [9], the authors investigated the asymptotic behavior of solutions of second order nonlinear impulsive differential equations. The authors [14] obtained the results of Razumikhin and Krasovskii stability of impulsive stochastic delay systems via uniformly stable function method. However, the behavior results of solutions for impulsive differential equations mainly focus on first order impulsive differential equations and second order impulsive differential equations. For higher order impulsive differential equations, the asymptotic behavior and stability of solutions has been little discussed.

Motivated by the above discussions, this paper is to study the asymptotic behavior of n order impulsive differential equations. By constructing appropriate Lyapunov functions and impulsive technique, the bounded property and zero convergence of every solution are obtained, which means that impulsive effects play an essential role in the behavior of nth order impulsive differential equations. Meanwhile, an example with simulations is provided to demonstrate the applicability of our results.

The rest of this paper is organized as follows. nth order impulsive differential equations is presented in Section 2. In Section 3, asymptotic behavior results of nth order impulsive differential equations are derived. A numerical example is given to demonstrate our results in Section 4 and Section 5 concludes the paper.

2. Model Description and Preliminaries

Consider the following nth order impulsive differential equations

$$\begin{cases} x^{(n)}(t)+p(t)x(t)=0, t\geq 0, t\neq t_k, \\ x^{(i)}(t_k^+)=I_{ik}(x^{(i)}(t_k)), \\ i=0,1,\cdots,n-1, k=1,2,\cdots, \end{cases} \tag{1}$$

where $t_k, k = 1,2,\cdots$ are impulsive moments satisfying $0=t_0<t_1<\cdots<t_k<t_{k+1}<\cdots, \lim\limits_{k\to+\infty} t_k=+\infty$,

$x^{(i)}(t_k^+) = \lim_{h\to 0^+} \frac{x^{(i-1)}(t_k+h) - x^{(i-1)}(t_k^+)}{h}$, $x^{(i)}(t_k) = \lim_{h\to 0^-} \frac{x^{(i-1)}(t_k+h) - x^{(i-1)}(t_k)}{h}$, $p(t)$ is continuous and differentiable function on $[0,+\infty)$, $I_{ik}(\cdot)$ are continuous in R and there exist positive numbers d_k such that $|I_{ik}(x)| \le d_k|x|$.

By a solution $x = x(t)$, we mean a real function on $[0,+\infty)$ satisfies that $x^{(n)}(t) + p(t)x(t) = 0$ at each point $t \in [0,+\infty)$ with the possible exception of the points $t \neq t_k$ and $x^{(i)}(t_k^+) = I_{ik}(x^{(i)}(t_k))$ for any t_k.

Lemma 1. ([9]) Assume that $l(t)$ is a continuous function on $[0,+\infty)$. If there exist positive numbers v_k, $k = 1, 2, \cdots$ and constants $\gamma > 0$, $b > 0$, c such that for $t_{k+1} - t_k \le \gamma$, $\ln v_k + \int_{t_k}^{t_{k+1}} l(s)ds \le -b$ and

$$\ln v_k + \int_{t_k}^{t_k+\xi} l(s)ds \le c, \forall \xi \in [0,\gamma] \tag{2}$$

then

$$\ln h(t) \le -\frac{b}{\gamma}t + \max\{b+c, 0\}, t > 0, \tag{3}$$

where $h(t) = \prod_{0\le t_k<t} v_k \exp(\int_0^t l(s)ds)$.

Lemma 2. ([9]) Assume that $l(t)$ is a continuous function on $[0,+\infty)$. If there exist positive numbers v_k, $k = 0, 1, \cdots$ and $i \in \{1, 2, \cdots\}$ and $\omega > 0, c > 0$ such that for $t_{k+i} = t_k + \omega, v_{k+i} = v_k$, $l(t+\omega) = l(t)$ and

$$\ln(\prod_{t\le t_k<t+\omega} v_k) + \int_t^{t+\omega} l(s)ds \le -b, \forall t \ge 0, \tag{4}$$

then

$$\ln h(t) \le -\tfrac{b}{\omega}t + b + \max_{t\in[0,\omega]}\{|\ln(\prod_{0\le t_k<t} v_k)|\} + \omega \max_{t\in[0,\omega]}\{|l(t)|\}, t > 0, \tag{5}$$

where $h(t) = \prod_{0\le t_k<t} v_k \exp(\int_0^t l(s)ds)$.

3. Asymptotic Behavior Analysis

In this section, some criteria on asymptotic behavior results of (1) are established under Lyapunov functions and impulsive technique.

Theorem 1. If there exists $M > 0$ such that $\alpha_n \le M$, $n = 1, 2, \cdots$, then every solution $x(t)$ of (1) is bounded, where $\alpha_n = d_1^2 \cdot \cdots \cdot d_{n-1}^2 \exp(\int_0^{t_n} \tilde{p}(s)ds)$, $\tilde{p}(s) = \max\{2, 1 + p^2(s)\}$.

Proof. For every solution $x(t)$ of (1), we construct a Lyapunov function

$$V(t) = \sum_{i=0}^{n-1}[x^{(i)}(t)]^2 \tag{6}$$

In view of system (1), we have

$$\begin{aligned} V'(t) &= 2\sum_{i=0}^{n-1}[x^{(i)}(t)x^{(i+1)}(t)] = 2x^{(n-1)}(t)[-p(t)x(t)] + 2\sum_{i=0}^{n-2}[x^{(i)}(t)x^{(i+1)}(t)] \\ &\le \sum_{i=0}^{n-1}\{[x^{(i)}(t)]^2 + [x^{(i+1)}(t)]^2\} + p^2(t)[x^{(n-1)}(t)]^2 + x^2(t) \\ &= \sum_{i=0}^{n-2} 2[x^{(i)}(t)]^2 + [1+p^2(t)][x^{(n-1)}(t)]^2 \le \tilde{p}(t)V(t). \end{aligned} \tag{7}$$

When $t = t_k$, we have

$$\begin{aligned} V(t_k^+) &= \sum_{i=0}^{n-1}[x^{(i)}(t_k^+)]^2 = \sum_{i=0}^{n-1}[I_{ik}(x^{(i)}(t_k))]^2 \\ &\le \sum_{i=0}^{n-1} d_k^2[x^{(i)}(t_k)]^2 = d_k^2 V(t_k). \end{aligned} \tag{8}$$

For $t \in [0, t_1]$, by (7), it yields that

$$V(t) \le V(0)\exp(\int_0^t \tilde{p}(s)ds). \tag{9}$$

Thus

$$V(t_1) \le V(0)\exp(\int_0^{t_1} \tilde{p}(s)ds). \tag{10}$$

For $t \in [t_1, t_2]$, we have

$$\begin{aligned} V(t) &\le V(t_1^+)\exp(\int_{t_1}^t \tilde{p}(s)ds) \\ &\le d_1^2 V(t_1)\exp(\int_{t_1}^t \tilde{p}(s)ds) \\ &\le V(0)d_1^2\exp(\int_0^{t_1} \tilde{p}(s)ds + \int_{t_1}^t \tilde{p}(s)ds) \\ &= V(0)d_1^2\exp(\int_0^t \tilde{p}(s)ds). \end{aligned} \tag{11}$$

By induction, for $t \in [t_{n-1}, t_n]$, we conclude that

$$\begin{aligned} V(t) &\le V(0)d_1^2 \cdot \dots \cdot d_{n-1}^2 \exp(\int_0^t \tilde{p}(s)ds) \\ &\le V(0)d_1^2 \cdot \dots \cdot d_{n-1}^2 \exp(\int_0^{t_n} \tilde{p}(s)ds) \\ &= V(0)\alpha_n. \end{aligned} \tag{12}$$

Since sequence $|\alpha_n| \le M$, we can conclude that every solution $x(t)$ of (1) is bounded.

Theorem 2. If $\lim_{n\to\infty} \alpha_n = 0$, then every solution $x(t)$ of (1) satisfies $\lim_{t\to\infty} x(t) = 0$, where $\alpha_n = d_1^2 \cdot \dots \cdot d_{n-1}^2 \exp(\int_0^{t_n} \tilde{p}(s)ds)$, $\tilde{p}(s) = \max\{2, 1 + p^2(s)\}$.

Theorem 3. If there exist $a > 0$, $\overline{M} > 0$ such that $p(t) \ge a$ and $\beta_n \le \overline{M}$, $n = 1, 2, \cdots$, then every solution $x(t)$ of (1) is bounded, where $\beta_n = d_1^2 \cdot \dots \cdot d_{n-1}^2 \exp(\int_0^{t_n} \bar{p}(s)ds)$, $\bar{p}(s) = \max\{2, 2p(s) + \frac{p'(s)}{p(s)}\}$.

Proof. For every solution $x(t)$ of (1), we construct a Lyapunov function

$$V(t) = p(t)x^2(t) + \sum_{i=1}^{n-1} [x^{(i)}(t)]^2 \tag{13}$$

In view of system (1), we have

$$\begin{aligned} V'(t) &= p'(t)x^2(t) + 2p(t)x'(t)x(t) + 2\sum_{i=1}^{n-1} [x^{(i)}(t)x^{(i+1)}(t)] \\ &= p'(t)x^2(t) + 2p(t)x'(t)x(t) + 2\sum_{i=1}^{n-2} [x^{(i)}(t)x^{(i+1)}(t)] + 2x^{(n-1)}(t)[-p(t)x(t)] \\ &\le p'(t)x^2(t) + p^2(t)(t)x^2(t) + [x'(t)]^2 + \sum_{i=0}^{n-2} \{[x^{(i)}(t)]^2 + [x^{(i+1)}(t)]^2\} + [x^{(n-1)}]^2 + p^2(t)x^2(t) \\ &= [2p^2(t) + p'(t)]x^2(t) + 2\sum_{i=1}^{n-1} [x^{(i)}(t)]^2 \\ &= (2p(t) + \tfrac{p'(t)}{p(t)})p(t)x^2(t) + 2\sum_{i=1}^{n-1} [x^{(i)}(t)]^2 \\ &\le \bar{p}(t)V(t). \end{aligned} \tag{14}$$

According to the proof of Theorem 1, we can conclude that every solution $x(t)$ of (1) is bounded.

Theorem 4. If there exists $a > 0$ such that $p(t) \ge a$ and $\lim_{n\to\infty} \beta_n = 0$, then every solution $x(t)$ of (1) satisfies $\lim_{t\to\infty} x(t) = 0$, where $\beta_n = d_1^2 \cdot \dots \cdot d_{n-1}^2 \exp(\int_0^{t_n} \bar{p}(s)ds)$, $\bar{p}(s) = \max\{2, 2p(s) + \frac{p'(s)}{p(s)}\}$.

Theorem 5. If there exist $c_1 > 0$, $c_2 \ge 0$ such that $\ln h_1(t) \le -c_1 t + c_2$, then every solution $x(t)$ of (1) is globally uniformly exponentially stable, where $h_1(t) = \prod_{0 \le t_k < t} d_k^2 \exp(\int_0^t \tilde{p}(s)ds)$, $\tilde{p}(s) = \max\{2, 1 + p^2(s)\}$.

Proof. For every solution $x(t)$ of (1), we construct a Lyapunov function

$$V(t) = \sum_{i=0}^{n-1} [x^{(i)}(t)]^2 \tag{15}$$

Based on the proof of Theorem 1, for $t \in [t_{n-1}, t_n]$, we have

$$\begin{aligned} V(t) &\le V(0)d_1^2 \cdot \dots \cdot d_{n-1}^2 \exp(\int_0^t \tilde{p}(s)ds) \\ &= V(0) \prod_{0 \le t_k < t} d_k^2 \exp(\int_0^t \tilde{p}(s)ds) \\ &= V(0)h_1(t) \le V(0)e^{c_2}e^{-c_1 t}, \end{aligned} \tag{16}$$

which implies that every solution $x(t)$ of (1) is globally uniformly exponentially stable.

Theorem 6. If there exist $a > 0$, $c_3 > 0$, $c_4 \ge 0$ such that $p(t) \ge a$ and $\ln h_2(t) \le -c_3 t + c_4$, then every solution $x(t)$ of (1) is globally uniformly exponentially stable, where $h_2(t) = \prod_{0 \le t_k < t} d_k^2 \exp(\int_0^t \bar{p}(s)ds)$, $\bar{p}(s) = \max\{2, 2p(s) + \frac{p'(s)}{p(s)}\}$.

By using Lemma 1 and Lemma 2, together with Theorem 5 and Theorem 6, we can obtain the following practical theorems.

Theorem 7. If there exist $\bar{\gamma}_1 > 0$, $\bar{b}_1 > 0$, $\bar{c}_1$ such that for $k = 1, 2, \cdots$, $t_{k+1} - t_k \le \bar{\gamma}_1$, $\ln d_k^2 + \int_{t_k}^{t_{k+1}} \tilde{p}(s)ds \le -\bar{b}_1$ and

$$\ln d_k + \int_{t_k}^{t_k + \bar{\xi}_1} \tilde{p}(s)ds \le \bar{c}_1, \forall \bar{\xi}_1 \in [0, \bar{\gamma}_1] \tag{17}$$

then every solution $x(t)$ of (1) is globally uniformly exponentially stable, where
$\tilde{p}(s) = \max\{2, 1 + p^2(s)\}$.

Theorem 8. If there exist $a > 0$, $\bar{\gamma}_2 > 0$, $\bar{b}_2 > 0$, $\bar{c}_2$ such that for $p(t) \ge a$, $t_{k+1} - t_k \le \bar{\gamma}_2$, $\ln d_k^2 + \int_{t_k}^{t_{k+1}} \bar{p}(s)ds \le -\bar{b}_2$

and

$$\ln d_k^2 + \int_{t_k}^{t_k+\bar{\xi}_2} \bar{p}(s)ds \leq \bar{c}_2, \forall \bar{\xi}_2 \in [0, \bar{\gamma}_2], \tag{18}$$

then every solution $x(t)$ of (1) is globally uniformly exponentially stable, where

$\bar{p}(s) = \max\{2, 2p(s) + \frac{p^{'}(s)}{p(s)}\}.$

Theorem 9. If there exist $i \in \{1, 2, \cdots\}$ and $\omega > 0, \tilde{b}_1 > 0$ such that $t_{k+i} = t_k + \omega, d_{k+i} = d_k, \tilde{p}(t+\omega) = \tilde{p}(t)$ and

$$\ln(\prod_{t \leq t_k < t+\omega} d_k^2) + \int_t^{t+\omega} \tilde{p}(s)ds \leq -\tilde{b}_1, \forall t \geq 0, \tag{19}$$

then every solution $x(t)$ of (1) is globally uniformly exponentially stable, where

$\tilde{p}(t) = \max\{2, 1 + p^2(t)\}.$

Theorem 10. If there exist $i \in \{1, 2, \cdots\}$ and $a > 0$ $\omega > 0, \tilde{b}_2 > 0$ such that $p(t) \geq a$ $t_{k+i} = t_k + \omega, d_{k+i} = d_k$, $\bar{p}(t+\omega) = \bar{p}(t)$ and

$$\ln(\prod_{t \leq t_k < t+\omega} d_k^2) + \int_t^{t+\omega} \bar{p}(s)ds \leq -\tilde{b}_2, \forall t \geq 0, \tag{20}$$

then every solution $x(t)$ of (1) is globally uniformly exponentially stable, where

$\bar{p}(s) = \max\{2, 2p(s) + \frac{p^{'}(s)}{p(s)}\}.$

4. Numerical Simulations

This section presents an example to demonstrate theoretical results for asymptotic behavior of nth order impulsive differential equations.

Example 1. Consider the following second order impulsive differential equations:

$$\begin{cases} x^{''}(t) + p(t)x(t) = 0, t \geq 0, t \neq t_k, \\ x(t_k^+) = d_k x(t_k)), x^{'}(t_k^+) = d_k x^{'}(t_k)), k = 1, 2, \cdots, \end{cases} \tag{21}$$

where $p(t)$ is periodic function with periodic $\omega = 2$, and

$$p(t) = \begin{cases} 0, t \in [0, 0.5), \\ 2, t \in [0.5, 1], \end{cases}$$

the impulsive effects with periodic 1 are defined by

$t_{k+2} = t_k + 1$, $t_1 = 0.2$, $d_{2k} = 0.2$, $d_{2k+1} = 1.25$, $k = 0, 1, \cdots$. By computation, we have

$$\tilde{p}(t) = \begin{cases} 2, t \in [0, 0.5), \\ 5, t \in [0.5, 1] \end{cases}$$

and

$$\ln(\prod_{t \leq t_k < t+2} d_k^2) + \int_t^{t+2} \tilde{p}(s)ds < 0, \forall t \geq 0. \tag{22}$$

It follows from Theorem 9 that every solution $x(t)$ of (1) is globally uniformly exponentially stable. Figure 1 depicts impulsive sequence with period 1. Figure 2 depicts state trajectory $x(t)$ of system (21).

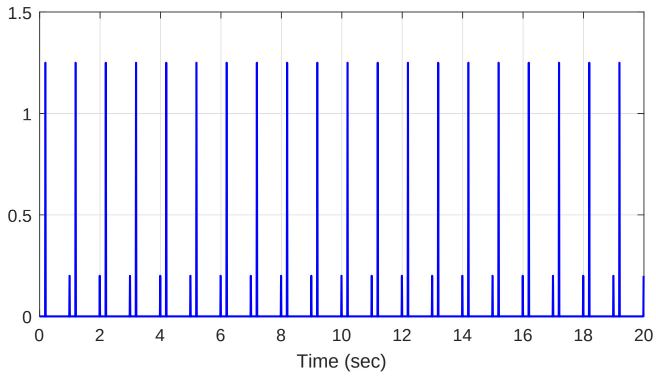

Figure 1. *Impulsive sequence with periodic 1.*

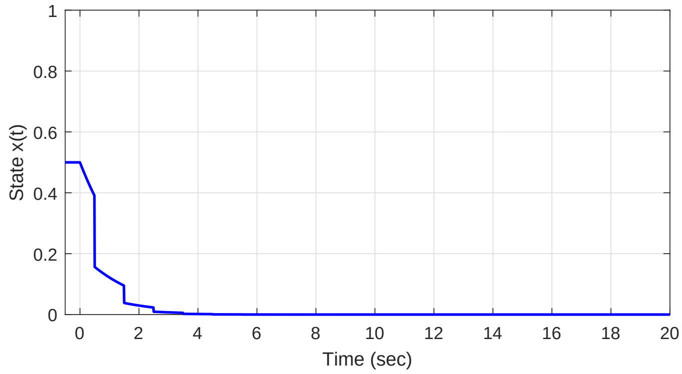

Figure 2. *The state trajectory $x(t)$ of (21).*

5. Conclusions

In this paper, asymptotic behavior of n order impulsive differential equations has been studied. By method of impulsive technique and Lyapunov function, some new behavior criteria have been derived. Finally, a standard example package illustrate that the new results are practical. Our future work will focus on the neural networks with impulse and consensus of multi-agent systems with impulse.

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Conflicts of Interest

The authors declare no conflicts of interest.

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