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Extension of Power Mean and Logarithms Mean

Yisa Anthonio[1], Abimbola Abolarinwa[2], Abdullai Abdurasid[1], Kamal Rauf[3], Michael Adeniyi[1], Adeyinka Ogunsanya[4, *], Christian Iluno[1]

[1]Department of Pure and Applied Mathematics, Lagos State University of Science and Technology, Ikorodu, Nigeria
[2]Department of Mathematics, University of Lagos, Lagos, Nigeria
[3]Department of Mathematics, University of Ilorin, Ilorin, Nigeria
[4]Department of Statistics, University of Ilorin, Ilorin, Nigeria

Email address:
ogunsanyaadeyinka@yahoo.com (A. Ogunsanya)
[*]Corresponding author

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Abstract: Logarithms are indispensable in the revision of mathematics which are basic components tools in the theory of mathematical analysis. Logarithms have playing acute fundamental role in the study of the properties of power and arithmetic means as well as inequalities of Logarithms with their bound. This paper shows the properties of logarithms mean, power mean, arithmetic mean, Harmonic mean, geometric mean and later we use Minkowski's inequality and Hölder's inequality to establish the modified means. In the paper, we obtained the generalization of power mean, logarithms mean, arithmetic mean, Harmonic mean and geometric mean. The methodology adopted are Minkowski's inequality and Hölder's inequality to establish some means of order α of two distincts. These inequalities further generalize some existing results. This research work also demonstrated the importance of the Minkowski's inequality and Hölder's inequality over existing arithmetic mean, Harmonic mean and geometric mean and further extend the generalization to weighted logarithms mean. Hence, this article distinguished some present results on power mean, logarithms means and acquired more robust means by engaging modified Minkowski's inequality and Hölder's inequality with some ordinary theorems. The modified Minkowski's inequality on power and logarithms mean further extends the generalized weighted logarithms mean of order α of two distincts.

Keywords: Extension of Logarithms Mean, Power Mean, Arithmetic Mean, Geometric Mean, Harmonic Mean, Minkowski's Inequality and Hölder's Inequality

1. Introduction

The application of different forms of means has attracted many researchers in the field of mathematics, which has led to several extension of logarithms means, power mean, geometric mean and harmonic mean. This was investigated by Pal et al [1] The importance of generalization of inequalities involving logarithmic mean and power mean cannot be overemphasized as stated by E. B. Leach and M. C. Sholander [2]. The Extensions and generalizations of inequalities involving the logarithmic mean, power mean and their applications have been extensive studied in the literature [see 2-5].

Tung-Po Lin [6] assumed p as the least value and q the greatest value for all value of positive numbers x and y respectively of the logarithmic mean and the power mean of order α of two distinct. Given

$$K_p < L_r(x,y) < K_q$$

is consistent for all distinct nonnegative real numbers x and y.

Tung-Po Lin [6] demostrated in his work that at q=0 and p=1/3 are the optimal solution for the above order which may satisfy any non-negative real numbers x and y. H. Alzer [7] further established that the inequalities from Tung-Po Lin [6] is better compared to existing inequalities.

Alzer [7- 8] studied a form of generalized logarithmic mean which is a special case of the Stolarsky mean. Define thus:

$$L_r(x,y)=\begin{cases}\frac{r}{r+1}\frac{x^{r+1}-y^{r+1}}{x^r-y^r}, r\neq 0,-1, x\neq y\\ \frac{Inx-Iny}{x-y}, r=0, x\neq y\\ xy\frac{x-y}{Inx-Iny}, r=0, x\neq y\\ x, x=y\end{cases} \quad (1)$$

so that $\vartheta(x,y)=L_0(x,y)$.

In 1995 J. Sa'ndor, obtained certain refinements for inequalities involving means, results attributed to Carlson [9]; Leach and Sholander; Alzer; Sándor; and Vamanamurthy and Vuorinen.

See $[10-12]$, theses papers gave an easy proof compare to Tung-Po Lin$'$s of a more general result. S. Furuichi and H. R. Moradi [13] introduced weighted logarithmic mean and its inequalities among weighted means were demonstrated. Farissi et al [14] applied the standard Hermite-Hadamard inequalities to logarithms means. Y. M. Chu and W. F. Xia [15] proved that harmonic mean is greater than logarithms mean i.e. $H(\varepsilon,\tau)\geq L_p(\varepsilon,\tau)$. More precisely and the authors obtained the classes of functions f and $h_\alpha, \alpha\in\mathbb{R}$ such that

$$In\, f_\alpha(t)-\frac{h_\alpha(t)}{\left(t^{\frac{1}{\alpha}+I}\right)^\alpha}>0, t>1.$$

We shall consider lower bound value τ and upper bound value value ε such that

$$M_\tau<\vartheta<M_\varepsilon$$

is hold for all difference non- negative real numbers τ and ε, where

$$(\varepsilon,\tau)=\lim_{(x,y)\to(\varepsilon,\tau)}\frac{\varepsilon-\tau}{In(\varepsilon)-In(\tau)}=\begin{cases}\varepsilon\ if\ \varepsilon=\tau\\ \frac{\varepsilon-\tau}{In(\varepsilon)-In(\tau)}\ otherwise\end{cases} \quad (2)$$

and

$$\sqrt{\varepsilon\tau}\leq\vartheta(\varepsilon,\tau)\leq\left(\frac{\varepsilon^{\frac{1}{n}}+\tau^{\frac{1}{n}}}{2}\right)^n\leq\frac{\varepsilon+\tau}{2}\varepsilon,\tau>0. \quad (3)$$

Are the logarithmic mean and power mean of order n of two different non- negative real numbers ε and τ respectively.

Given two numbers (ε,τ), then the logarithmic mean $\vartheta(\varepsilon,\tau)$ is less than the arithmetic mean and the generalized mean with exponent one third but greater than the geometric mean, otherwise the numbers are the same, in which all three means are equal to the numbers. We will use the above inequality in our later discussion.

The Mean value theorem of differential calculus is given as follows:

Given mean value theorem, there exists a value ξ in the interval between ε and τ where the function f derivative f' equals the gradient of the secant line: exists

$$\xi\in(\varepsilon,\tau): f'(\xi)=\frac{f(\varepsilon)-f(\tau)}{\varepsilon-\tau}$$

We get logarithmic mean value of ξ by substituting by ln for f and same for its corresponding derivative of natural logarithm:

$$\frac{1}{\xi}=\frac{\ln(\varepsilon)-\ln(\tau)}{\varepsilon-\tau}$$

and solving for ξ:

$$\xi=\frac{\varepsilon-\tau}{\ln(\varepsilon)-\ln(\tau)}$$

The logarithmic mean of two positive nnumbers can also be taken as the area under an exponential function that is exponential curve.

In light of this, we present an integral insight using Minkowski$'$s inequality and H$\ddot{o}$lder$'$s inequality to generalize logarithmic mean. We further extend the generalization to weighted logarithmic mean.

Preliminaries Result

The following form of logarithm mean inequality is required:

The prove of above logarithmic mean and power mean are as follows to establish our main result;

Theorem 1: Suppose τ and ε are non negative real functions defined on an open interval (τ,ε). Then, the following inequality holds:

$$\int_0^1\tau^t\varepsilon^t dt\leq\int_0^1 t\tau+(1+t)\varepsilon dt \quad (4)$$

Then, we have

$$\vartheta(\varepsilon,\tau)=\frac{\varepsilon-\tau}{ln(\varepsilon)-ln(\tau)}\leq\frac{1}{2}\tau+\frac{1}{2}\varepsilon=\frac{(\tau+\varepsilon)}{2} \quad (5)$$

equality holds if $\tau=\varepsilon$.

Proof:

Integrate both sides of above inequality yields

$$\vartheta(\tau,\varepsilon)=\int_0^1\tau^{1-t}\varepsilon^t\,\mathrm{d}t\leq\int_0^1 t\tau+(1-t)\varepsilon\,\mathrm{d}t$$
$$=\frac{\tau}{\ln\left(\frac{\varepsilon}{\tau}\right)}\left(\frac{\varepsilon}{\tau}-1\right)=\frac{\tau}{\ln\left(\frac{\varepsilon}{\tau}\right)}\left(\frac{\varepsilon-\tau}{\tau}\right)\leq\frac{1^2}{2}\tau+\left(1-\frac{1^2}{2}\right)\varepsilon-\frac{0^2}{2}\tau+\left(0-\frac{0^2}{2}\right)\varepsilon$$
$$=\frac{\varepsilon-\tau}{\ln\left(\frac{\varepsilon}{\tau}\right)}\leq\frac{1}{2}\tau+\varepsilon-\frac{1}{2}\varepsilon$$

$$\vartheta(\varepsilon,\tau)=\frac{\varepsilon-\tau}{\ln(\varepsilon)-\ln(\tau)}\leq\frac{1}{2}\tau+\frac{1}{2}\varepsilon=\frac{(\tau+\varepsilon)}{2}. \quad (6)$$

Further simplification, if right hand side of (6) that is $\frac{\varepsilon-\tau}{\ln(\varepsilon)-\ln(\tau)}$ is replaced with $\tau=\gamma_1^{\frac{1}{2}}\gamma_2^{\frac{1}{2}}$ and $\varepsilon=\gamma_3^{\frac{1}{2}}\gamma_4^{\frac{1}{2}}, \gamma>0$

gets

$$\frac{1}{4}\left(\gamma_1^{\frac{1}{2}}+\gamma_2^{\frac{1}{2}}+\gamma_3^{\frac{1}{2}}+\gamma_4^{\frac{1}{2}}\right) \tag{7}$$

To extend it to finitely many numbers for any positive integer n and positive numbers $x_1, x_2, \dots x_{2^n}$ we have

$$(x_1, x_2, \dots x_{2^n})^{\frac{1}{2n}} \le \frac{1}{2n}(x_1, x_2, \dots, x_{2^n}) \tag{8}$$

If we set ζ equal to ε and the remaining it's equal to τ (where $0 < \zeta < 2^n$) yields

$$\varepsilon^{\frac{\zeta}{2^n}}\tau^{1-\frac{\zeta}{2^n}} - \frac{\zeta}{2^n}\varepsilon - \left(1+\frac{\zeta}{2^n}\right)\tau \le \infty \tag{9}$$

The above inequality showed arithmetic-geometric mean inequality:

The area interpretation made it simple to derive certain basic logarithmic mean properties. Since the exponential function is a case of monotonic function, then the integral over an interval ε and τ of length 1 is bounded by the same values.

Then, we want to obtain useful integral inequality mean they are as follows:

Theorem 2: Suppose τ and ε are non negative real functions defined on an open interval (τ, ε). Then, the following inequality holds:

$$\frac{1}{(t+\varepsilon)(\tau+t)} \le \infty$$

$$\vartheta(\varepsilon, \tau) = \int_0^\infty \frac{dt}{(t+\varepsilon)(\tau+t)} = \frac{1}{\varepsilon-\tau} \times \int_0^\infty \left(\frac{dt}{(t+\varepsilon)} - \frac{dt}{(t+\tau)}\right)$$

$$\frac{1}{\varepsilon-\tau} \times \log\left(\frac{(t+\tau)}{(t+\varepsilon)}\bigg|_0^\infty\right) = \frac{1}{\varepsilon-\tau} \times \log\tau - \log\varepsilon \le \infty \tag{10}$$

The homogeneous function of the integral operator can be adopted in the mean operator, that is

$$\vartheta(\rho\varepsilon, \rho\tau) = \rho\vartheta(\varepsilon, \tau)$$

The above inequality, as with other means, we have $\vartheta(\rho\varepsilon, \rho\tau) = \rho(\varepsilon, \tau)$ for $\rho > 0$. In particular,

$$\vartheta(\varepsilon, \tau) = \varepsilon\vartheta(1; \tfrac{\tau}{\varepsilon}) = \varepsilon\tau\vartheta(\tfrac{1}{\varepsilon}, \tfrac{1}{\tau}).$$

If the mean value theorem is considered, the above inequality can be generalized to $n+1$ variables by divided differences for derivative of the logarithm.

Hence, it gives

$$\vartheta(\varepsilon_0, \dots, \varepsilon_n) = \sqrt[-n]{(-1)^{(n+1)} n \ln(\varepsilon_0, \dots, \varepsilon_n)}$$

where $\ln(\varepsilon_0, \dots, \varepsilon_n)$ denote a divided difference of the logarithm. The Arithmetic mean form (4) is given, if the right side of (4) is replaced with $t\tau^2 + (1-t)\varepsilon^2$, then integrate with respect to t. We have

$$\vartheta(\varepsilon^2, \tau^2) = \int_0^1 t\tau^2 + (1-t)\varepsilon^2 dt = \frac{t^2}{2}\tau^2 + (t - \frac{t^2}{2})\varepsilon^2 = \frac{\tau^2+\varepsilon^2}{2}$$

$$\frac{\vartheta(\varepsilon^2, \tau^2)}{\vartheta(\varepsilon, \tau)} = \frac{\frac{\tau^2+\varepsilon^2}{2}}{\frac{\varepsilon+\tau}{2}} = \frac{\varepsilon+\tau}{2}$$

The Geometric mean form (4) is given, if the right side of (4) is replaced with $t\tau^2 + (1-t)\varepsilon^2$, then integrate with respect to t. We have

$$\vartheta\left(\frac{1}{\varepsilon}, \frac{1}{\tau}\right) = \int_0^1 t\frac{1}{\varepsilon} + (1-t)\frac{1}{\tau} dt = \frac{t^2}{2\varepsilon} + \frac{(t-\frac{t^2}{2})}{\tau}\bigg|_0^1 = \frac{\frac{\varepsilon+\tau}{2}}{\frac{\varepsilon+\tau}{2\varepsilon\tau}} = \varepsilon\tau$$

$$Hence, \sqrt{\frac{\vartheta(\varepsilon,\tau)}{\vartheta\left(\frac{1}{\varepsilon},\frac{1}{\tau}\right)}} = \sqrt{\frac{\frac{\varepsilon+\tau}{2}}{\frac{\varepsilon+\tau}{2\varepsilon\tau}}} = \sqrt{\varepsilon\tau}$$

The Harmonic mean form (4) is given, if also the right side of (4) is replaced with $t\tau^2 + (1-t)\,\varepsilon^2$, then integrate with respect to t. We have

$$\vartheta\left(\frac{1}{\varepsilon^2}, \frac{1}{\tau^2}\right) = \int_0^1 t\frac{1}{\varepsilon^2} + (1-t)\frac{1}{\tau^2} dt = \frac{\varepsilon^2+\tau^2}{2\varepsilon^2\tau^2}$$

$$\therefore \frac{\vartheta\left(\frac{1}{\varepsilon},\frac{1}{\tau}\right)}{\vartheta\left(\frac{1}{\varepsilon^2},\frac{1}{\tau^2}\right)} = \frac{\frac{\varepsilon+\tau}{2\varepsilon\tau}}{\frac{\varepsilon^2+\tau^2}{2\varepsilon^2\tau^2}} = \frac{2}{\frac{1}{\varepsilon}+\frac{1}{\tau}}$$

2. Main Results

Theorem 3: Suppose (X, ζ, μ) be a finite space and $f: \varepsilon \to \mathbb{R}$ be a μ-integrable defined the integral arithmetic mean (the conditional expectation of the random variable f in which case it is denoted) $\sum f$ in probability theory) by

$$M_i(f, \mu) = \frac{1}{\mu(\varepsilon)}\int_X f d\mu \tag{11}$$

and suppose $g: X \to \mathbb{R}$ be a μ-integrable function.

(a) If f is a convex function given on an interval I that include the image of g and $\psi: I \to \mathbb{R}$ is a function such that $\psi \in \partial f(\varepsilon)$ for every $\tau \in I$ and $\psi^o g$ and $g(\psi^o g)$ are $\mu-$ integrable functions, then, the following inequality holds:

$$0 \le M_i(f^o g) - f(M_1(g)) \le M_i(g(\psi^o g)) - M_1(g)M_1(\psi^o g) \; and \tag{12}$$

(b) if f is concave, then, the (12) holds in the reverse direction.

Proof:

Taking the left hand side of (12) that is $M_i(f^o g) - f(M_1(g))$ and since $f^o g$ is μ-integrable function equality holds for f strictly convex if and only if g is constant $\mu-$ almost everywhere. Let $M_1(g) \in I$ if otherwise $h = M_1(g) - g(or\ -h)$ will be a strictly positive function whose integral will be zero that is, $\psi: I \to \mathbb{R}$ satisfies.(a) for every ε in the interior of I. If $M_1(g)$ is also in the interior of I, then

$$f(M) \le f(g(\varepsilon))$$

If

$$q(\varepsilon) = \int_b^{\varepsilon} f(\varepsilon,\tau)d\tau,$$
$$p(\varepsilon) = \int_b^{\infty} d\tau\left(\int_b^{y} f(\varepsilon,\tau)^{a}d\tau\right)^{\frac{1}{a}},$$
$$r(\varepsilon) = \int_b^{\infty} d\tau\left(\int_\tau^{\infty} f(\varepsilon,\tau)^{a}d\tau\right)^{\frac{1}{\alpha}} \textit{ and the inequality} \quad (13)$$
$$\int_b^{\infty} q(\varepsilon)d\varepsilon \le r(\varepsilon)^{\alpha}$$

holds and sufficient by Theorem (1) that

$$\int_b^{\infty} q(\varepsilon)g(\varepsilon)d\varepsilon\ \forall g(\varepsilon) \quad (14)$$

such that

$$\int_\tau^{\infty} q(\varepsilon)^{b}d\varepsilon \le 1$$

$$\int_b^{\infty} q(\varepsilon)g(\varepsilon)d\varepsilon = \int_b^{\infty} d\tau\left(\int_b^{x} f(\varepsilon,\tau)d\tau\right)g(\varepsilon)d\varepsilon$$

$$= \int_b^{\infty} d\tau\left(\int_\varepsilon^{\tau} f(\varepsilon,\tau)g(\varepsilon)d\varepsilon\right) = \int_b^{\infty} d\tau\left(\int_\tau^{\infty} f(\varepsilon)^{\alpha}d\varepsilon\right)^{\frac{1}{\alpha}}\left(\int_\tau^{\infty} g(\varepsilon)^{\beta}d\varepsilon\right)^{\frac{1}{\beta}}$$

$$\le \int_b^{\infty} d\tau\left(\int_\tau^{\varepsilon} f(\varepsilon,\tau)d\varepsilon\right)^{\frac{1}{\alpha}}$$

using (2.4) yields

$$\int_b^{\infty} \left(\int_b^{\varepsilon} f(\varepsilon,\tau)d\tau\right)^{\alpha}d\varepsilon \le \left(\int_b^{\infty} d\tau\left(\int_\tau^{\infty} f(\varepsilon,\tau)^{\alpha}d\varepsilon\right)^{\frac{1}{\alpha}}\right)^{\alpha}$$

Hence,

$$\left(\int_b^{\infty} \left(\int_b^{\varepsilon} f(\varepsilon,\tau)d\varepsilon\right)^{\alpha}d\tau\right)^{\frac{1}{\alpha}} \le \int_b^{\infty} \left(\int_\tau^{\infty} f(\varepsilon,\tau)d\varepsilon\right)^{\frac{1}{\alpha}}d\tau.$$

The (14) can be written as

$$\int_b^{\infty} q(\varepsilon)g(\varepsilon)d\varepsilon = \int_b^{\infty} \left(\int_\varepsilon^{\infty} f(\varepsilon,\tau)d\tau\right)g(\varepsilon)d\varepsilon = \int_b^{\infty} d\tau\left(\int_\varepsilon^{\infty} g(\varepsilon)f(\tau,\varepsilon)d\varepsilon\right)$$
$$\le \int_b^{\infty} d\tau\left(\int_\varepsilon^{\infty} g(\varepsilon)^{\frac{1}{\beta}}d\varepsilon\right)^{\beta}\left(\int_\varepsilon^{\infty} f(\varepsilon,\tau)^{\alpha}d\varepsilon\right)^{\frac{1}{\alpha}} = \int_b^{\infty} d\varepsilon\left(\int_\varepsilon^{\infty} f(\varepsilon,\tau)^{\alpha}d\varepsilon\right)^{\frac{1}{\alpha}} \quad (15)$$

Hence,

$$\int_b^{\infty} \left(\int_\varepsilon^{\infty} f(\varepsilon,\tau)d\tau\right)^{a}d\varepsilon \le \int_b^{\infty} \left(\int_b^{\tau} f(\varepsilon,\tau)^{\alpha}d\varepsilon\right)^{\frac{1}{\alpha}}d\tau$$

and

$$\left(\int_b^{\infty} \left(\int_\varepsilon^{\infty} f(\varepsilon,\tau)d\tau\right)^{\alpha}d\varepsilon\right)^{\frac{1}{a}} \le \int_b^{\infty} \left(\int_b^{\tau} f(\varepsilon,\tau)^{\alpha}d\varepsilon\right)^{\frac{1}{\alpha}}d\tau$$

Theorem 4: let $q_1(\varepsilon), q_2(\varepsilon), \ldots, q_n(\varepsilon)$ *are real functions such that* $q_n(\varepsilon) \ge 0$ *and* $n \ge 1$. $\sum_i^{n-1} r_i$*=1 and* $r_i(\tau) \ge 0$ *we have*

$$\prod_{i=1}^{n} q_i^{r_i}(\varepsilon) \le \sum_{i=1}^{n} r_i(\varepsilon)q_i(\varepsilon) \le \left(\sum_{i=1}^{n} r_i(\varepsilon)q_i(\varepsilon)^{n}\right)^{\frac{1}{n}} \quad (16)$$

Proof:
If

$$q_1(\varepsilon) \times q_2(\tau) = \left(\frac{q_1(\varepsilon)+q_2(\varepsilon)}{2}\right)^2 - \left(\frac{q_1(\varepsilon)-q_2(\varepsilon)}{x}\right)^2 \le \left(\frac{q_1(\varepsilon)+q_2(\varepsilon)}{2}\right)^2$$

$$q_1(\varepsilon) \times q_2(\varepsilon) \times q_3(\varepsilon) \times q_4(\varepsilon) \le \left(\frac{q_1(\varepsilon)+q_2(\varepsilon)}{2}\right)^2\left(\frac{q_3(\varepsilon)+q_4(\varepsilon)}{2}\right)^2$$

$q_1(\varepsilon) \times q_2(\varepsilon) \times q_3(\varepsilon) \times \ldots q_n(\varepsilon)$ and the arithmetic mean ϖ taking $2^m - n$time then we have

$$q_1(\varepsilon) \times q_2(\varepsilon) \times q_3(\varepsilon) \times \ldots q_n(\varepsilon) \times \varpi^{2m-n} \leq \left(\frac{q_1(\varepsilon)+q_2(\varepsilon)+\cdots+q_n(\varepsilon)+(2^m-n)\varpi}{2^m}\right)^{2^m}$$

Therefore,

$$(q_1(\varepsilon) \times q_2(\varepsilon) \times q_3(\varepsilon) \times \ldots q_n(\varepsilon))^{\frac{1}{n}} \leq \varpi.$$

We observed that geometric mean is always less than the arithmetic mean unless all the $q'(\varepsilon)s$ are equal.
Inequality (18) can also be written in order n of positive numbers $q_1(\varepsilon), \ldots q_n(\varepsilon)$ with weights $\varepsilon_1(\varepsilon), \ldots \varepsilon(\tau)$ as below

$$G_n(\chi; s) = (\textstyle\sum_{i=1}^{n} \tau(\varepsilon) q_i(\varepsilon)^n)^{\frac{1}{n}}, n \neq 0 \; and \; G_n(\chi; s) = \prod_{i=1}^{n} q_i^{r_i}(\varepsilon), n = 0 \tag{17}$$

This reduces to generalize weighted logarithmic mean if $n = 0$ and refined result.

$$L_n(\chi; s) = \int_\Omega \prod_{1=\mathrm{n}}^{n} q_i^{\varepsilon_i}(\varepsilon) d\chi(\tau) \tag{18}$$

The positive numbers $q_1(\tau), \ldots, q_n(\tau)$ is generalized weighted logarithmic mean by

$$F_n(\chi; s) = \int_\Omega G_n(\chi; s) d\chi(\tau) \tag{19}$$

The means $F_n(\chi; q)$ are increasing sequences in n. If is well known that the power mean $G_n(\chi; s)$ is increasing sequence in n and in (19) the same is true for $F_n(\chi; s)$.
Theorem 5: Let τ, η be positive numbers with $\tau + \eta = 1$ such that $n \geq 0$. Then

$$G_n(\chi; s^{\eta+1}, q^{\tau+1}) \leq G_n(\chi; s)^{\eta+1} G_n(\chi; q)^{\tau+1} = G_n(\chi; s)^{\eta} G_n(\chi; s) G_n(\chi; q)^{\eta} G_n(\chi; q) \tag{20}$$

$$where \; (s^{\eta+1}, q^{\tau+1}) = \left(s_1^{\eta+1} q_1^{\tau+1}, s_2^{\eta+1} q_2^{\tau+1}, s_3^{\eta+1} q_3^{\tau+1}, \ldots, s_{n-1}^{\eta+1} q_{n-1}^{\tau+1}, s_n^{\eta+1} q_n^{\tau+1}\right)$$

Proof:
If $n > 0$, then integral $H\ddot{o}$lder's inequality holds:

$$\begin{aligned} &G_n(\chi; s^{\eta+1}, q^{\tau+1}) = \int_\Omega f_n(s^{\eta+1}, q^{\tau+1}; \tau) d\chi(\tau) \\ &= \int_\Omega \left(\textstyle\sum_{1=\mathrm{n}}^{\Psi} \varepsilon_i (s_i^{\eta+1} q_i^{\tau+1})^n\right)^{\frac{1}{n}} d\chi(\tau) \leq \int_\Omega \left(\textstyle\sum_{1=\mathrm{n}}^{\Psi} \tau \, s_i^n\right)^{\frac{\eta+1}{n}} \left(\textstyle\sum_{1=\mathrm{n}}^{\Psi} \tau_i q_i^n\right)^{\frac{\tau+1}{n}} d\chi(\tau) \\ &\leq G_n(\chi; s)^{\eta} G_n(\chi; s) G_n(\chi; q)^{\eta} G_n(\chi; q) \end{aligned} \tag{21}$$

Theorem 6: Let τ, η be positive numbers with $\tau + \eta = 1$ such that $n \geq 0$. Then

$$G_n(\varepsilon; s^{\eta+1}, q^{\tau+1}) \leq G_n(; (\eta+1)s + (\tau+1)q) \tag{22}$$

$$where \; ((\eta+1)s + (\tau+1)q) \leq ((\eta+1)s_1 + (\tau+1)q_1, \ldots, (\eta+1)s_{n-1} + (\tau+1)q_{n-1}, (\eta+1)s_n + (\tau+1)q_n)$$

Proof:
The arithmetic-geometric inequality if $n \neq 0$

$$G_n(\chi; s^{\eta+1}, q^{\tau+1}) = \int_\Omega f_n(s^{\eta+1}, q^{\tau+1};) d\chi(\tau) = G_n(\chi; (\eta+1)s + (\tau+1)q) \tag{23}$$

In above means, G_n gives

$$G_n(s_1 + q_1, s_2 + q_2) \leq G_n(s_1 + s_2) + G_n(q_1 + q_2) \; if \; n \geq 1$$

and

$$G_n(s_1 + q_1, s_2 + q_2) \geq G_n(s_1 + s_2) + G_n(q_1 + q_2) \; if \; if \; n \leq 1$$

We shall adapt Minkowski 's inequality to prove arithmetic mean.
Theorem 7: Let $n \geq 1$ such that

$$G_n(\chi; s^{\eta+1} + q^{\tau+1}) \leq G_n(\chi; s^{\eta+1}) + G_n(\chi; q^{\tau+1}) \tag{24}$$

while for $n \leq 1$ the inequality is reversed.
Proof:
If $n \geq 1$, using Minkowski's inequality yields:

$$G_n(\chi; s^{\eta+1} + q^{\tau+1}) = \int_\Omega f_n(s^{\eta+1} + q^{\tau+1}; \chi) d\chi(\varepsilon) = \int_\Omega \left(\sum_{i=1}^{\Psi} \varepsilon_i (s_i^{\eta+1} + q_i^{\tau+1})^n\right)^{\frac{1}{n}} d\chi(\varepsilon)$$
$$\leq G_n(\chi; s)^\eta G_n(\chi; s) G_n + (\chi; q)^\eta G_n(\chi; q) \tag{25}$$

If $r < 0$ the reverse of above results is obtained, that is Minkowski′s inequality.

3. Conclusion

The study's findings distinguished some present results on power mean and logarithms means and acquired more robust means by engaging modified Minkowski's inequality and Hölder's inequality with some ordinary theorems. The results protracted and generalized some earlier results in literature.

Conflicts of Interest

The authors declare that they have no competing interests.

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